ZESTAW EKSTREMA II

zad. 3. b) \( f(x, y) = x^2 + y^4 - 2xy + 1 \)

\[
\frac{\partial f}{\partial x} = 2x - 2y \\
\frac{\partial f}{\partial y} = 4y^3 - 2x
\]

\[
\frac{\partial^2 f}{\partial x^2} = 2 \\
\frac{\partial^2 f}{\partial y^2} = 12y^2 \\
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -2
\]

\[
\begin{bmatrix}
2 & -2 \\
-2 & 12y^2
\end{bmatrix}
\]

\[
\begin{cases}
2x - 2y = 0 \\
4y^3 - 2x = 0
\end{cases}
\]

\[
\begin{cases}
x = y \\
2x(x^2 - \frac{1}{2}) = 0
\end{cases}
\]

\((xy) \in \left\{ (0, 0) , \left( \frac{1}{2}, \frac{1}{2} \right) , \left( -\frac{1}{2}, -\frac{1}{2} \right) \right\} \]

1° \( P_1 = (0, 0) \)

\[
\begin{bmatrix}
2 & -2 \\
-2 & 0
\end{bmatrix}
\]

\( d_1 = 2 \)

\( d_2 = -4 \)

Nie ma ekstremum
\[
\begin{bmatrix}
2 & -2 \\
-2 & 6
\end{bmatrix}
\begin{array}
d_1 = 2 \\
d_2 = 8
\end{array}
\]

Minimum lokalne w \( P_1, P_2 \)

\[f(x,y) = e^{x^2-y} (5 - 2xy)\]

\[\frac{\partial f}{\partial x} = e^{x^2-y} \cdot 2x (5 - 2xy) + 2e^{x^2-y} = e^{x^2-y} (10x - 4x^2 + 2xy + 2)\]

\[\frac{\partial f}{\partial y} = e^{x^2-y} \cdot (2x - 5y) + e^{x^2-y} \cdot 2y = e^{x^2-y} (2x - 5y + 1)\]

\[
\begin{cases}
40x - 4x^2 + 2xy + 2 = 0 \\
2x - 6 - y + 1 = 0 \Rightarrow y = 2x - 4
\end{cases}
\]

\[
\begin{cases}
y = 2x - 4 \\
-4x^2 + 10x + 2(2x - 4) + 2 = 0
\end{cases}
\]

\[
\begin{cases}
y = 2x - 4 \\
-4x^2 + 10x + 2x^2 - 8x + 2 = 0
\end{cases}
\]

\[
\begin{cases}
y = 6 \\
x = -1
\end{cases}
\]

\[P = (-1, -6)\]
\[
\frac{\partial^2 f}{\partial y^2} = e^{x+y} \left( y + 5 - 2x - 1 \right) + e^{x+y} \left( -1 \right) = e^{x+y} \left( y + 5 - 2x - 2 \right)
\]
\[
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( e^{x+y} \left( y + 5 - 2x - 1 \right) \right) = e^{x+y} \left( y + 5 - 2x - 1 \right) + e^{x+y} \left( -2 \right) = e^{x+y} \left( y + 5 - 2x - 3 \right)
\]
\[
\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( e^{x+y} \left( y + 5 - 2x - 1 \right) \right) = e^{x+y} \left( y + 5 - 2x - 1 \right) + e^{x+y} \left( -2x + 6 \right) = e^{x+y} \left( y + 5 - 2x + 6 \right)
\]
\[
\frac{\partial^2 f}{\partial x \partial y} \left( -1, -6 \right) = e^{x+y} \left( y + 5 - 2x - 1 \right) \bigg|_{x=-1, y=-6} = e^{-1} \left( -6 + 5 - 2 + 2 \right) = -\frac{1}{e}
\]
\[
\frac{\partial^2 f}{\partial y^2} \left( -1, -6 \right) = e^{x+y} \left( y + 5 - 2x - 1 \right) \bigg|_{x=-1, y=-6} = e^{-1} \left( -6 + 5 - 2 + 2 \right) = -\frac{1}{e}
\]
\[
\frac{\partial^2 f}{\partial x \partial y} \left( -1, -6 \right) = e^{x+y} \left( y + 5 - 2x - 1 \right) \bigg|_{x=-1, y=-6} = e^{-1} \left( -6 + 5 - 2 + 2 \right) = -\frac{1}{e}
\]
\[
\begin{bmatrix}
-10e^7 & 2e^7 \\
2e^7 & -5e^7
\end{bmatrix}
\begin{bmatrix}
10 \\
2 - 5
\end{bmatrix}
= 0 = e^7 \cdot
\begin{bmatrix}
10 + 2 \\
12 - 5
\end{bmatrix}
\]
\[
d_1 = -10e^7 < 0 \\
d_2 = 6e^7 > 0
\]
Maksimumm
Problem 1. \( f(xy) = x^2 + y^2 \)

\( P_0 = (1, 1) \)

\[ v = \left[ -6, \frac{2}{3} \right] \]

\[ \hat{v} = \frac{-2}{\sqrt{37}} \]

\[ Df(1, 1) = \left( f(1+t+\frac{2}{3}t, 1+t+\frac{2}{3}t) - f(1, 1) \right) \]

\[ = \lim_{t \to 0} \frac{1 - \frac{5}{6}t + \frac{9}{36}t^2 + 1 + \frac{8}{36}t + \frac{16}{36}t^2 - 1}{t} = \lim_{t \to 0} \frac{\frac{2}{3}t + t^2}{t} = \]

\[ = \lim_{t \to 0} \left( \frac{2}{3} + t \right) = \frac{2}{3} \]

Problem 3.

c) \( z(xy) = \ln(xy) - x^2 - y^2 \)

\[ \frac{\partial f}{\partial x} = \frac{1}{xy} - 2x \]

\[ \frac{\partial f}{\partial y} = \frac{1}{xy} - 2y \]

\[ \frac{1}{xy} - 2x = 0 \]

\[ \frac{1}{xy} - 2y = 0 \]

\[ x = y \]

\[ \frac{1}{2x} - 2x = 0 \]

\[ 1 - 4x^2 = 0 \]

\[ x = \frac{1}{2}, \sqrt{\frac{1}{2}} \]

\[ x = y = \frac{1}{2}, x = y = -\frac{1}{2} \]
\[
\frac{\partial^2 f}{\partial x^2} = -\frac{1}{(xy)^2} - 2 \\
\frac{\partial^2 f}{\partial y^2} = -\frac{1}{(xy)^2} - 2 \\
\frac{\partial^2 f}{\partial y \partial x} = \frac{1}{(xy)^2} = -\frac{1}{(xy)^2} \\
\begin{bmatrix}
\frac{1}{(xy)^2} & -2 \\
2 & \frac{1}{(xy)^2} \\
-2 & \frac{1}{(xy)^2} \end{bmatrix}
\]

\[N\left(\frac{1}{2}, \frac{1}{2}\right) = M\left(-\frac{1}{2}, -\frac{1}{2}\right) = \begin{bmatrix} -3 & -1 \\
-1 & -3 \end{bmatrix} \]

\[d_1 = -3 < 0 \quad \text{Maximum} \]

\[d_2 = 8 > 0 \quad \text{Minimum} \]

d) \[f(x,y) = x - 2y + \ln\sqrt{x^2 + y^2} + 3\arctan\frac{y}{x}\]

\[\frac{\partial f}{\partial x} = 1 + \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{1}{x} \cdot 2x + \frac{3}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{y}{x} = \frac{1}{x} \cdot \frac{x^2 + 2y^2}{x^2 + y^2} \]

\[\frac{\partial f}{\partial y} = -2 + \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{3x}{x^2 + y^2} + \frac{2(x^2 + xy) + 3y}{x(x^2 + y^2)} - \frac{2y^2 + 3y}{x^2 + y^2} \]

\[\begin{cases} 
x^2 + y^2 - 3y = 0 \\
2x^2 + 2xy + 3y = 0 \end{cases} \]
\[ \begin{align*}
2x^2 - y^2 + 3y &= 0 \quad 12 \\
2x^2 - 2y^2 + xy + 3x &= 0 \quad 14
\end{align*} \]

\[ 2x - 6y + 3x = 0 \]

\[ x = 2y \]

\[ 2x^2 - 2x = 0 \]

\[ x = 0 \quad \text{or} \quad x = 1 \]

\[ 2xy = 0 \quad \text{or} \quad x = y = 1 \]

\[ \frac{\partial^2 f}{\partial x^2} = \left( \frac{2(x+3y)^2 - (x^2+y^2+3y)^2}{(x^2+y^2)^2} \right) = \frac{2x^2 + 2y^2 + y^2 - 2x^2 - 2y^2 - 2x^2 + 6xy}{(x^2+y^2)^2} = \]

\[ = \frac{2x^2 + 6xy}{(x^2+y^2)^2} \]

\[ \frac{\partial^2 f}{\partial y^2} = \left( \frac{(1+xy)(x^2+y^2) - 2y(-2x^2-2y^2+2y+3x)}{(x^2+y^2)^2} \right) = \]

\[ = \frac{x^2+y^2-4y^2-4y^3+4x^2y+4y^3-2y^2-6xy}{(x^2+y^2)^2} \]

\[ = \frac{x^2-4y^2+4y^3+4x^2y+4y^3-2y^2-6xy}{(x^2+y^2)^2} \]

\[ = \frac{x^2-4y^2-6xy}{(x^2+y^2)^2} \]
\[
\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \left( \frac{-2x^2 - 2y^2 + y + 3x}{(x^2 + y^2)^2} \right) = 
\]
\[
\frac{(8-4y)(x^2+y^2) - 2x(-2x^2 - 2y^2 + 4y + 3x)}{(x^2 + y^2)^2} = 
\]
\[
\frac{3x^2 + 3y^2 - 4x^2 - 4xy^2 + 4x^3 + 4xy^2 - 2xy + 6x^2}{(x^2 + y^2)^2} = 
\]
\[
\frac{3y^2 - 3x^2 - 2xy}{(x^2 + y^2)^2} = 
\]
\[
\frac{3y^2 - 3x^2 - 2xy}{(x^2 + y^2)^2} 
\]

\[
\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{-2x^2 - 2y^2 + y + 3x}{(x^2 + y^2)^2} \right) = 
\]
\[
\frac{(2x-3y)(x^2+y^2) - 2y(2x^2 - y + 3)}{(x^2 + y^2)^2} = 
\]
\[
\frac{2x^2 + 2y^2 - 3x^2 - 3y^2 - 2xy - 2y^3 - 2y + 6y^2}{(x^2 + y^2)^2} = 
\]
\[
\frac{3y^2 - 3x^2 - 2xy}{(x^2 + y^2)^2} 
\]

\[
\left[ \begin{array}{cc}
\frac{3}{2} & -1 \\
-1 & \frac{3}{2}
\end{array} \right] 
\]

\[
d_1 = \frac{9}{2} > 0 \\
d_2 = \frac{3}{2} < 0 
\]

up mode, break electro.

\[f(x, y, z) = x + y + 2z \]
\[g(x, y, z) = z^2 + x = 1 \]
\[q(x, y) = x^2 + y^2 + z^2 - 1 = 0 \]

\[L(x, y, z) = x + y + 2z + \lambda(z^2 + x^2 - 1) \]

\[\frac{\partial L}{\partial x} = 1 + 2\lambda x \]
\[\frac{\partial L}{\partial z} = 2 + 2\lambda x \]
\[\frac{\partial L}{\partial y} = 1 + 2\lambda y \]
\[\frac{\partial L}{\partial \lambda} = z^2 + x^2 - 1 \]
\[ 1 + 2\lambda x = 0 \Rightarrow x = -\frac{1}{2\lambda} \]

\[ 1 + 2\lambda y = 0 \Rightarrow y = -\frac{1}{2\lambda} \]

\[ 2 + 2\lambda z = 0 \Rightarrow z = -1 \]

\[ 2 + 2\lambda z = 0 \Rightarrow z = -1 \]

\[ \frac{2}{x^2} + \frac{4}{y^2} = \frac{1}{\lambda} \]

\[ \lambda^2 = \frac{2}{y^2} \quad \text{for} \quad \lambda \neq 0 \]

\[ X = \pm \sqrt{\frac{2}{\lambda}} \quad \lambda \neq 0 \]

\[ \frac{\partial x}{\partial x} = 2x \quad \frac{\partial y}{\partial y} = 2y \quad \frac{\partial z}{\partial z} = 2z \]

\[ \frac{\partial^2}{\partial x^2} = 2 \lambda \quad \frac{\partial^2}{\partial y^2} = 2 \lambda \quad \frac{\partial^2}{\partial z^2} = 2 \lambda \]

\[ \frac{\partial^2}{\partial xy} = \frac{\partial^2}{\partial yx} = \frac{\partial^2}{\partial xz} = \frac{\partial^2}{\partial zx} = \frac{\partial^2}{\partial yz} = 0 \]

\[ (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \]

\[ \begin{bmatrix} 0 & -\frac{\sqrt{2}}{\sqrt{2}} & \frac{\sqrt{2}}{\sqrt{2}} & \frac{\sqrt{2}}{\sqrt{2}} \\ -\frac{\sqrt{2}}{\sqrt{2}} & 2 & 0 & 0 \\ 0 & 2 & 0 & 2 \sqrt{2} \\ -\frac{2 \sqrt{2}}{\sqrt{2}} & 0 & 0 & 2 \sqrt{2} \end{bmatrix} \]

\[ H_2 = -\frac{8 \sqrt{2}}{6} - \frac{8 \sqrt{2}}{6} < 0 \]

\[ H_3 = -2 \cdot \frac{8 \sqrt{2}}{6} - 2 \sqrt{2} < 0 \]

\[ \text{Minimum} \]

\[ (\frac{4}{\sqrt{2}}, \frac{4}{\sqrt{2}}, 2, -2) \]

\[ \begin{bmatrix} 0 & \frac{2 \sqrt{2}}{\sqrt{2}} & \frac{2 \sqrt{2}}{\sqrt{2}} & \frac{2 \sqrt{2}}{\sqrt{2}} \\ \frac{2 \sqrt{2}}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \sqrt{2} \end{bmatrix} \]

\[ H_2 = \frac{8 \sqrt{2}}{6} + \frac{8 \sqrt{2}}{6} > 0 \]

\[ H_3 = -\frac{2 \sqrt{2}}{6} \cdot 8 \sqrt{2} - 2 \sqrt{2} (8 \sqrt{2} \sqrt{8} \sqrt{4}) < 0 \]

\[ \text{Maximum} \]
\( f(x, y) = xy \ln(xy) \)

\[
\frac{\partial f}{\partial x} = y \ln(xy) + xy \frac{1}{xy} \Rightarrow x + y > 0
\]
\[
\frac{\partial f}{\partial y} = x \ln(xy) + xy \frac{1}{xy} \Rightarrow x + y > 0
\]

\[
\begin{cases}
y \ln(xy) + \frac{xy}{x + y} = 0 \\
x \ln(xy) + \frac{xy}{x + y} = 0
\end{cases}
\]

\[
y \ln(xy) = x \ln(xy)
\]
\[
\ln(xy)^2 = \ln(x+y)^x
\]
\[
(x+y)^y = (x+y)^x \quad \text{Mod}(x+y) / \log(x+y)
\]

\[
\int \frac{(x+y)^x}{(x+y)^y - 1} \, dx = 0
\]
\[
x + y = 0
\]

\[
y = x
\]

\[
x \ln(2x) + \frac{x}{2x} = 0
\]
\[
x (\ln 2x + \frac{1}{2}) = 0
\]
\[
x = 0 \quad \text{or} \quad \ln 2x + \frac{1}{2} = 0
\]
\[
x = \frac{1}{e^2}
\]
\[
y = x = \frac{1}{e^2}
\]

\[
\frac{\partial^2 f}{\partial x^2} = \frac{1}{(xy)^2} + \frac{y(2y-x)-x(2y-y)}{x+y} = \frac{x}{xy} + \frac{y}{x+y} = \frac{x}{xy} + \frac{y}{x+y} = \frac{2y^2 + xy}{(x+y)^2}
\]
\[
\frac{\partial^2 f}{\partial y^2} = \frac{1}{(xy)^2} + \frac{x(2x-y)-y(2x-x)}{x+y} = \frac{x}{x+y} + \frac{y}{x+y} = \frac{2x^2 + xy}{(x+y)^2}
\]
\[
\frac{\partial^2 f}{\partial x \partial y} = \frac{1}{xy} - \ln(xy) + \frac{x}{x+y} = \frac{2x^2 + xy}{(x+y)^2} - \ln(xy) + \frac{x}{x+y} = \frac{2x^2 + xy}{(x+y)^2}
\]
\[ D(a \times g = \frac{4}{25}) \]

\[ M_{\text{minimum}} \]

**h)** \( f(x_1, \ldots, x_n) = x_1 + \ldots + x_n + \frac{4}{x_1} + \ldots + \frac{1}{x_n} \)

\[
\frac{\partial^2 f}{\partial x_i^2} = \frac{2}{x_i^3} - \frac{1}{x_i^2} = 0 \quad \text{for } x_i = 1, x_i = -1
\]

\[
\begin{bmatrix}
\frac{2}{x_1^3} & 0 & \cdots & 0 \\
0 & \frac{2}{x_2^3} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{2}{x_n^3}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{d}_1 \\
\text{d}_2 \\
\vdots \\
\text{d}_n
\end{bmatrix} = \begin{bmatrix}
1 & 2 \\
2 & 1 \\
\vdots & \vdots \\
\text{max}
\end{bmatrix}
\]

\[
(x_1 = 1, x_1 = -1) \Rightarrow \text{min}
\]

\[
(-1, -1, \ldots, -1) \Rightarrow \text{max}
\]

**f)** \( f(x,y,z) = x + y + z + \frac{4}{x} + \frac{1}{y} + \frac{1}{z} \)

\[ d_1 = \frac{1}{x} \\
\frac{d_2}{dx} = \frac{1}{y} \\
\frac{d_3}{dx} = \frac{1}{z} \]

\[ f(x,y,z) = x + y + z + \frac{4}{x} + \frac{1}{y} + \frac{1}{z} \]

\[ f(0,0,0) = 0 \]
\[ f(x,y,z) = x^4 - y^3 + 2z^3 - 2x^2 + 6y^2 - 3z \]

\[ \frac{dx}{dx} = 4x^3 - 4x \]
\[ \frac{dy}{dx} = -3y^2 + 12y = -3y(y-4) \]
\[ \frac{dz}{dz} = 6z^2 - 6z = 6z(z-1) \]

\[ \frac{dx}{dy} = \frac{24}{2} = \frac{24}{dy} \quad \frac{dy}{dz} = \frac{24}{dz} \quad \frac{dz}{dx} = \frac{24}{dx} = 0 \]

\[ \frac{dx^2}{dy} = 12x - 4 \]
\[ \frac{dy}{dz} = -6y + 12 \]
\[ \frac{dz}{dx} = 12z - 6 \]

\[ (1, 0, 0) \quad (1, 0, 0) \quad (-1, 0, 0) \]
\[ (0, 4, 0) \quad (1, 4, 0) \quad (-1, 4, 0) \]
\[ (0, 4, 1) \quad (1, 4, 1) \quad (-1, 4, 1) \]
\[ (0, 0, 1) \quad (1, 0, 1) \quad (-1, 0, 1) \]

\[ d_1 < 0 \quad d_2 > 0 \quad d_3 < 0 \]
\[ f(0,4,0) = 32 \]

Maximum:
\[ (1, 0, 0) \]
\[ \begin{bmatrix} -4 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & 6 \end{bmatrix} \]
\[ \text{det} = 0 \]
\[ d_1 < 0 \quad d_2 > 0 \quad d_3 > 0 \]
\[ \text{ niekon. brak ekstr. } \]

Minimum:
\[ (1, 0, 0) \]
\[ \begin{bmatrix} 12 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & 6 \end{bmatrix} \]
\[ \text{det} = 0 \]
\[ d_1 > 0 \quad d_2 < 0 \]
\[ d_3 < 0 \]
\[ \text{ niekon. brak ekstr. } \]
\[
\left( 1,1,1 \right) \begin{bmatrix}
8 & 0 & 0 \\
0 & -12 & 0 \\
0 & 0 & 6
\end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \geq 0 \quad \left( 1,0,1 \right) \begin{bmatrix}
8 & 0 & 0 \\
0 & 12 & 0 \\
0 & 0 & 6
\end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \geq 0
\]

\[
\left( 0,1,0 \right) \begin{bmatrix}
8 & 0 & 0 \\
0 & -12 & 0 \\
0 & 0 & 6
\end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \geq 0
\]

\[
f(1,0,1) = -2
\]

\[
\left( -1,0,1 \right) \begin{bmatrix}
8 & 0 & 0 \\
0 & 12 & 0 \\
0 & 0 & 6
\end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \geq 0
\]

\[
f(-1,0,1) = -2
\]

\[
\text{ zad. 2. } L : (x,y,z) = (1,0,1) + t \cdot [2,1,4]
\]

\[
f(x,y) = x^2 + y^2
\]

\[
\n e = (2,1,4)
\]

\[
z - f(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0) (x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0) (y-y_0)
\]

\[
z - x_0^2 - y_0^2 = 2x_0(x-x_0) + 2y_0(y-y_0)
\]

\[
x_0^2 + y_0^2 \geq 2x_0y_0 - 2x_0 + 2y_0 + 2y_0^2 = 0
\]

\[
\bar{m} = (2x_0, -2y_0, 1)
\]

\[
\bar{m} \leq |m| = t \cdot \bar{e}
\]

\[
(2,1,4) = (-2x_0, -2y_0, t)
\]

\[
t = 4 \land x_0 = -\frac{4}{2} \land y_0 = -\frac{4}{8}
\]

\[
R = (\frac{4}{4}, -\frac{4}{8})
\]
\[ f(x, y, z) = \sin(x + y + z) + \sin(x) - \sin(y) - \sin(z) \]

\[
\frac{\partial f}{\partial x} = \cos(x + y + z) - \cos(x) = -\frac{1}{2} \sin(x + \frac{y + z}{2}) \sin\left(\frac{y + z}{2}\right) \\
\frac{\partial f}{\partial y} = \cos(x + y + z) - \cos(y) = -\frac{1}{2} \sin(y + \frac{x + z}{2}) \sin\left(\frac{x + z}{2}\right) \\
\frac{\partial f}{\partial z} = \cos(x + y + z) - \cos(z) = -\frac{1}{2} \sin(z + \frac{x + y}{2}) \sin\left(\frac{x + y}{2}\right)
\]

\[
\begin{align*}
\sin\left(x + \frac{y + z}{2}\right) \sin\left(\frac{y + z}{2}\right) &= 0 \\
\sin\left(y + \frac{x + z}{2}\right) \sin\left(\frac{x + z}{2}\right) &= 0 \\
\sin\left(z + \frac{x + y}{2}\right) \sin\left(\frac{x + y}{2}\right) &= 0
\end{align*}
\]

100
\[ x + \frac{y + z}{2} = k \pi \]

200
\[ x = y \]

1000
\[ \sin(x + y) \sin(x) = 0 \]

2000
\[ x = y = k \pi \]

z \in \mathbb{R}

Z = -k \pi

2000
\[ \frac{x + z}{2} = k \pi \]

000
\[ \frac{x + z}{2} = x + \frac{y + z}{2} \]

000
\[ 0 = \frac{x + y}{2} \]

\[ x = -y \]

\[ \sin(z) \sin(0) = 0 \]

1000
\[ z \in \mathbb{R} \]

\[ (x, y, z) \]

2000
\[ \frac{x + z}{2} = k \pi \]

2000
\[ x = y \]

\[ \sin(x + z) \sin(x) = 0 \]

1000
\[ x + z = k \pi \]

\[ x = -k \pi - z = x_{1,2} \in \mathbb{R} \]

\[ (x_{1,2}, y_{1,2}, z_{1,2}) \]
\[
\frac{\partial^2 f}{\partial x^2} = -\sin(x+yz) + \sin x \\
\frac{\partial^2 f}{\partial y^2} = -\sin(x+yz) + \sin y \\
\frac{\partial^2 f}{\partial z^2} = -\sin(x+yz) + \sin z \\
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial^2 f}{\partial y \partial z} = -\sin(x+yz)
\]

\[
(x_1, y_1, z_1, k_1)
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
d_1 = d_2 = d_3 = 0
\]

\[
(x_2, x_2, k_2, k_2)
\begin{bmatrix}
2 & 2 & 1 \\
2 & 1 & 2 \\
-2 & 1 & -2
\end{bmatrix}
\]

\[
k = 2k \Rightarrow \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \Rightarrow \text{j.w.}
\]

\[
k = 4k + 1
\]

\[
d_1, d_2, d_3 > 0
\]

\[
k = 4k + 3
\]

\[
d_1, d_2, d_3 < 0
\]

\[
(0, 0, 0) \text{ by } 80
\]